

# Nonassociativity, Malcev Algebras and String Theory

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**Dedicated to Bruno Zumino on the occasion of his 90th birthday**

## Abstract

Nonassociative structures have appeared in the study of D-branes in curved backgrounds. In recent work, string theory backgrounds involving three-form fluxes, where such structures show up, have been studied in more detail. We point out that under certain assumptions these nonassociative structures coincide with nonassociative Malcev algebras which had appeared in the quantum mechanics of systems with non-vanishing three-cocycles, such as a point particle moving in the field of a magnetic charge. We generalize the corresponding Malcev algebras to include electric as well as magnetic charges. These structures find their classical counterpart in the theory of Poisson-Malcev algebras and their generalizations. We also study their connection to Stueckelberg's generalized Poisson brackets that do not obey the Jacobi identity and point out that nonassociative string theory with a fundamental length corresponds to a realization of his goal to find a non-linear extension of quantum mechanics with a fundamental length. Similar nonassociative structures are also known to appear in the cubic formulation of closed string field theory in terms of open string fields, leading us to conjecture a natural string-field theoretic generalization of the AdS/CFT-like (holographic) duality.

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# 1 Introduction

Noncommutative and associative algebraic structures are hallmarks of quantum physics. In the Hilbert space formulation of quantum mechanics the linear operators, such as Hermitian operators corresponding to observables or generators of symmetries do not, in general, commute. Their actions on the states in the Hilbert space are however associative.

In the early period of quantum mechanics Pascual Jordan introduced the Jordan formulation of quantum mechanics with the hope of generalizing the underlying algebraic framework of quantum mechanics [1]. Since the commutator of two Hermitian operators is not Hermitian Jordan suggested working with the symmetric product that preserves Hermiticity. The linear operators acting in a Hilbert space form a commutative but nonassociative Jordan algebra under the symmetric product defined as one half the anti-commutator. Jordan algebras that can be realized as such are called special. Jordan's hope of finding a large class of Jordan algebras that are not special was dashed when he, von Neumann and Wigner classified all finite dimensional Jordan algebras [1]. They showed that with but one exception all finite dimensional Jordan algebras are special. The only Jordan algebra that has no realization in terms of associative operators acting on a vector space or Hilbert space with the Jordan product being one-half the anti-commutator is the algebra of  $3 \times 3$  Hermitian matrices over the octonions which was called the exceptional Jordan algebra and is denoted as  $J_3^\mathbb{O}$ .

It was shown in [2] that one can formulate quantum mechanics over the exceptional Jordan algebra,  $J_3^\mathbb{O}$ , satisfying all the axioms of quantum mechanics as formulated by von Neumann. This work answered in affirmative the question posed by von Neumann four decades earlier as to whether there exists a quantum mechanics whose projective geometry is non-Desarguan. The quantum mechanics defined by the exceptional Jordan algebra is referred to as octonionic quantum mechanics and it does not admit a Hilbert space formulation.

In the eighties Zelmanov proved that there do not exist any new non-special infinite dimensional Jordan algebras [3]. This rules out the existence of an infinite dimensional quantum mechanics that has no Hilbert space formulation. Hence the octonionic quantum mechanics stands alone as the only quantum mechanics that has no Hilbert space formulation. The fact that its geometry is non-Desarguan implies that it can not be embedded in a higher dimensional projective geometry [4].

The Jordan's formulation of quantum mechanics in terms of the symmetric Jordan product was generalized to a formulation in terms of quadratic Jordan algebras which extends to the octonionic quantum mechanics as well as to quantum mechanics over finite fields [5].

Remarkably, there exists a unique supergravity theory defined by the exceptional Jordan algebra [6] that is referred as the exceptional supergravity. In five space-time dimensions it describes the coupling of 14 Abelian vector multiplets to  $N = 2$  supergravity and is the largest of four magical Maxwell-Einstein supergravity theories (MESGT). The magical supergravity theories are the only unified 5D MESGTs whose scalar manifolds are symmetric spaces [6]. The exceptional supergravity has the groups of the  $E$ -series as its global symmetry group in five, four and three dimensions just like the maximal supergravity. However, the real forms of the noncompact global symmetry groups of the exceptional supergravity with eight supercharges are different from the real forms of the global symmetry groups of maximal supergravity with 32 supercharges in the respective dimensions. Global symmetry groups of the exceptional supergravity in these dimensions are defined by the underlying exceptional Jordan algebra  $J_3^{\mathbb{O}}$  over the real octonions  $\mathbb{O}$ . On the other hand the global symmetries of the maximal supergravity in the respective dimensions are defined by the split exceptional Jordan algebra  $J_3^{\mathbb{O}_S}$  over split octonions  $\mathbb{O}_S$ <sup>3</sup>.

Among the issues investigated by the founders of quantum mechanics was the question whether quantum mechanics requires the use of complex number field in its formulation. In a remarkable paper on the question of whether one can formulate quantum theory over the field of real numbers [8] Stueckelberg showed that one can generalize Poisson brackets in classical mechanics such that they no longer satisfy the Jacobi identity while still preserving the Liouville theorem and the fundamental  $H$ -theorem of Boltzmann. This result led him to ask the question whether there exists a nonlinear generalization of quantum mechanics that involves a fundamental length, corresponding to his generalized Poisson brackets.

Nambu [9] pointed out that there exists a natural generalization of the canonical Poisson bracket that involves a triple product, which is surprisingly non-trivial to quantize and seems to be relevant for the covariant formulation of M-theory [10]. Nambu's work was partly inspired by the work of Günaydin and Gürsey on the implications of extending the underlying division algebra

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<sup>3</sup>See the review lectures [7] and the references therein.

of quantum mechanics to octonions and the connection between color degrees of freedom of quarks and octonions [11].<sup>4</sup>

It was also argued that nonassociative structures appear naturally in closed string field theory [13, 14, 15]. Other physicists have persisted in being fascinated by the possibility of nonassociative structures (see the monographs [16], [17] for a more complete set of references on the subject).

In spite of all these efforts, it is fair to say that we still do not know what precise role intrinsic nonassociativity will play in a deeper understanding of fundamental physics, in general, and in the foundations of M/superstring theory, in particular. In this note, we point out the relevance of certain specific non-associative structures, such as Malcev algebras, to string theory and to quantum mechanical extensions of Stueckelberg's generalized Poisson brackets.

It is well known that the constant  $B$ -field background in string theory leads to a non-commutative but associative star product [18]. As a consequence, the Seiberg-Witten limit of open string field theory then leads to a noncommutative gauge theory [19, 20, 21, 22].

Following these classic results, it has been pointed out that the presence of a non-zero  $H = dB$  background can give rise to nonassociative star-products [23, 24]. More recently conformal field theories involving the three-form backgrounds have been studied in [25]. Thus nonassociative structure seem to find natural appearance in closed string theory after all. This is perhaps not surprising given the pioneering papers [13, 14, 15].

One of the main goals of this paper is to point out that in the case when  $H$  is constant, the physical situation is well known from the study of non-vanishing three-cocycles in quantum mechanics [26]. The canonical example is that of a non-relativistic electron moving in the field of a constant magnetic charge distribution [27], which is relevant for closed string theory in the constant  $H$  background [25]. As was pointed out in [28] the relevant nonassociative structure in the case of constant magnetic charge is the so called Malcev algebra [29, 30, 31, 32]. Thus the nonassociative algebraic structures which appear in closed string theory in the constant  $H$  background [25] are Malcev algebras. However, these structures need to be generalized in consistent string backgrounds with constant  $H$  fields, due to the presence of a

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<sup>4</sup> Note that the role of nonassociativity is not fundamental in the context of the Nambu bracket and that quantization can be carried out with standard Hilbert space methods [12].

non-trivial gravitational background. We also point out that these quantum structures have a natural classical limit, in which they are precisely related to the prescient work of Stueckelberg, reviewed in section 2.

Motivated by the discussion of strings in non-zero  $H$  background presented in section 3, in section 4 we present our main claim regarding the relevance of Malcev algebras in string theory and then in sections 4 and 5 we discuss the general structure of Malcev algebras in this context as well as the relation between nonassociative and associative structures. In particular, we generalize nonassociative Malcev algebras that naturally appear in the quantum mechanics of systems with non-vanishing three-cocycles, such as a point particle moving in the field of a magnetic charge, to include electric as well as magnetic charges. These results find their classical counterpart in the theory of Poisson-Malcev algebras [33], which we identify as the underlying nonassociative structures behind Stueckelberg's pioneering work on generalized Poisson brackets that do not obey the Jacobi identity. In section 6 we also point out the relation between Malcev algebras and certain special geometries. In the concluding section we stress that nonassociativity arises in the purely cubic form of Witten's open string field theory [34] as pointed out by Horowitz and Strominger [13, 14, 15], as long as one is not confined to the calculation of S-matrix elements [35, 36, 37]. The nonassociativity found in this context is relevant in understanding the appearance of closed string states [14, 15] in a purely open string field theory. In view of these facts, we close our paper with a brief discussion of the general role of nonassociativity in string theory (see also the discussion in [25]), and in particular, we discuss the role of nonassociativity in closed string field theory [14, 15], which leads us to conjecture a string-field theoretic generalization of the AdS/CFT-like (holographic) duality.

## 2 Stueckelberg's Generalization of Poisson brackets and Boltzmann's H-theorem

In this section we review Stueckelberg's generalization of Poisson brackets in classical statistical mechanics while preserving Liouville's theorem and Boltzmann's H-theorem that follows from it [8]. Let  $\xi^\alpha$  denote the coordinates in phase space with  $\alpha, \beta, \dots = 1, 2, \dots, 2n$ . Conservation of energy  $H = H(\xi(t))$

implies that

$$\dot{\xi}^\alpha(t) = \partial_t \xi^\alpha(t) = \Omega^{\beta\alpha}(\xi(t)) \partial_\beta H(\xi(t)) \quad (1)$$

where  $\Omega^{\alpha\beta}(\xi) = -\Omega^{\beta\alpha}(\xi)$  is a symplectic "metric" in phase space. The scalar density  $\mathbf{w}(\xi, t)$  of the Gibbs ensemble is positive semi-definite and satisfies the continuity equation

$$\int d^{2n}\xi \mathbf{w}(\xi, t) = 1 \quad (2)$$

$$\partial_t \mathbf{w}(\xi, t) + \partial_\alpha (\dot{\xi}^\alpha \mathbf{w}(\xi, t)) = 0. \quad (3)$$

Under coordinate transformations  $\xi^\alpha \rightarrow \xi'^\alpha$  in phase space  $\mathbf{w}$  transforms as

$$\mathbf{w} \implies \mathbf{w}'(\xi', t) = |Det(\partial_\alpha \xi'^\beta(\xi))| \mathbf{w}(\xi, t). \quad (4)$$

Let  $dV(\xi) = g(\xi) d^{2n}\xi$  be the invariant volume element in phase space (with  $g(\xi)$  representing the "density of volume"). Then the invariant "scalar of the density"  $\omega(\xi, t)$  is defined as

$$\omega(\xi, t) := \frac{\mathbf{w}}{g}(\xi, t) > 0. \quad (5)$$

Since  $\Omega_{\alpha\beta}$  plays the role of metric in phase space Stueckelberg argues that the unique choice for  $g$  is

$$g = |Det(\Omega^{\alpha\beta})^{-1/2}| > 0. \quad (6)$$

Then the continuity equation takes the form

$$\partial_t \omega + D_\alpha (\xi^\alpha \omega) = 0 \quad (7)$$

where  $D_\alpha = \partial_\alpha + \partial_\alpha(\log g)$ . Now the Liouville theorem states the "scalar of density" must remain constant as the system evolves

$$\frac{d}{dt} \omega(\xi(t), t) := \dot{\omega}(\xi, t) = (\partial_t \omega + \dot{\xi}^\alpha \partial_\alpha \omega)(\xi, t) = 0. \quad (8)$$

This implies

$$D_\alpha \dot{\xi}^\alpha = D_\alpha ((\partial_\beta H) \Omega^{\beta\alpha}) = (\partial_\alpha \partial_\beta H) \Omega^{\alpha\beta} + (\partial_\beta H) D_\alpha \Omega^{\alpha\beta} = 0 \quad (9)$$

which requires that the fundamental tensor  $\Omega^{\alpha\beta}$  satisfy

$$D_\alpha \Omega^{\alpha\beta} = q^\beta = 0. \quad (10)$$

In terms of the density  $\mathcal{W}^{\alpha\beta} = g \Omega^{\alpha\beta}$  it can be expressed as

$$\partial_\alpha \mathcal{W}^{\alpha\beta} = q^\beta = 0 \quad (11)$$

which has the form of the second set of Maxwell's equations with vanishing electric charges  $q^\alpha$ .

Thus the time evolution of an observable  $F(\xi)$  is given by

$$\dot{F}(\xi) = \dot{\xi}^\alpha \partial_\alpha F(\xi) = \partial_\beta H \Omega^{\beta\alpha} \partial_\alpha F \equiv \{H, F\} = -\{F, H\} \quad (12)$$

where  $\{F, H\}$  defines the generalized Poisson bracket. The Jacobian defined as

$$J(F, G, H) \equiv \{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} \quad (13)$$

does not vanish in general under the generalized Poisson bracket. A simple calculation yields

$$J(F, G, H) = -3(\partial_\alpha F)(\partial_\beta G)(\partial_\gamma H) (\Omega^{\rho[\alpha} \partial_\rho \Omega^{\beta\gamma]}) . \quad (14)$$

To have the Jacobi identity satisfied we must require

$$-3\Omega^{\rho[\alpha} \partial_\rho \Omega^{\beta\gamma]} \equiv q^{\alpha\beta\gamma} = 0 \quad (15)$$

which can be written in the form of the first set of Maxwell equations with vanishing magnetic charges

$$\partial_{[\alpha} \Omega_{\beta\gamma]} \equiv q_{\alpha\beta\gamma} = 0. \quad (16)$$

Only when the magnetic charges  $q_{\alpha\beta\gamma}$  vanish one can choose a polarization and bring the metric  $\Omega^{\alpha\beta}$  to the canonical form locally:

$$\begin{vmatrix} 0 & -1 & 0 & 0 & . \\ 1 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & -1 & . \\ 0 & 0 & 1 & 0 & . \\ . & . & . & . & . \end{vmatrix}$$

Note that the above structure discovered by Stueckelberg fits beautifully into the Poisson-Malcev algebra discussed in the mathematics literature [33].

An associative commutative algebra  $A$  over a field  $F$  is called a Poisson-Malcev algebra if it is endowed with a  $F$ -bilinear map called the Poisson-Malcev bracket  $\{, \}$ , which is antisymmetric  $\{f_1, f_2\} = -\{f_2, f_1\}$  and which satisfies the following conditions:

$$\{f_1 f_2, f_3\} = f_1 \{f_2, f_3\} + \{f_1, f_3\} f_2 \quad (17)$$

i.e. the Leibnitz identity and

$$J(f_1, f_2, \{f_1, f_3\}) = \{J(f_1, f_2, f_3), f_1\} \quad (18)$$

for all  $f_1, f_2, f_3 \in A$  where the Jacobian  $J$  is defined as usual

$$J(f_1, f_2, f_3) \equiv \{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\}. \quad (19)$$

We will see that this is precisely the classical limit of the quantum Malcev algebra to be discussed in section 4!

## 2.1 Generalized Poisson brackets, minimal length and the stringy uncertainty principle

The above analysis of Stueckelberg shows that Jacobi identity is not necessary to establish the H-theorem. Furthermore Stueckelberg suggests that to the generalized Poisson brackets involving "magnetic charges" there must correspond a quantum theory in which the observables are no longer linear operators. This observation may have been one of the reasons why Stueckelberg considered the possible extension of quantum mechanics involving nonlinear operators. We should point out that nonassociativity corresponds to a particular kind of nonlinearity<sup>5</sup> and we shall argue that Stueckelberg's attempts to find a nonlinear extension of quantum mechanics should be replaced by attempts to find a nonassociative extension of quantum mechanics. The minimum length that he introduced for a non-linear extension of quantum mechanics would then be related to nonassociativity.

In particular Stueckelberg considered quantum theories with a "critical length"  $\lambda_0$  such that uncertainties in the measurements of coordinates satisfy

$$(\Delta X)^2 \geq (\lambda_0)^2 \quad (20)$$

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<sup>5</sup>In particular, nonassociativity precludes simple tensoring of free systems and thus the physics of nonassociativity should be intrinsically interactive and therefore nonlinear.

and proposed modifying the minimum uncertainty relation in one dimensional case as follows<sup>6</sup>

$$(\Delta X)^2(\Delta P)^2 = \frac{\hbar^2}{4} \left( 1 - \frac{(\lambda_0)^2}{(\Delta X)^2} \right)^{-1} \quad (21)$$

which requires the modification of canonical commutation relations as

$$i[P, X] = \hbar \left( 1 - \frac{(\lambda_0)^2}{(\Delta X)^2} \right)^{-1/2}. \quad (22)$$

This relation is, at first, a bit different from the canonical form of the string uncertainty principle [38, 39], which usually reads as

$$\Delta X \Delta P \sim (1 + \alpha' \Delta P^2) \hbar. \quad (23)$$

However, if we formally expand the inverse  $\left( 1 - \frac{(\lambda_0)^2}{(\Delta X)^2} \right)^{-1}$  and use to first order  $\Delta X \sim \hbar \Delta P^{-1}$  we precisely get the stringy uncertainty relation! (This formal procedure relates  $\lambda_0$  with the string scale  $l_s$ , or equivalently with  $\alpha' \sim l_s^2$ .) Both the physical and mathematical underpinnings of the stringy uncertainty relation are still not clear, even though one can attempt an algebraic formulation in terms of generalized  $X$  and  $P$  commutators (see the review in [39]). However, even at the classical level, at least naively, one is faced with certain interesting issues: for example, one might say that by the Darboux theorem one can always redefine  $X$  and  $P$  so that the generalized Poisson bracket has the canonical  $\{X, P\} = 1$  form. However, the Darboux theorem holds if the Jacobi identity is satisfied! Another problem with this reasoning is that  $X$  is constrained to be not smaller than  $\lambda_0$  (i.e. the string length scale  $l_s$ ). Thus the Poisson bracket between different  $X$ s has to be nontrivial to reflect this fact. (It is interesting that Stueckelberg does not comment about the  $[X_i, X_j]$  commutator because he is working in one dimension, which is trivial from the point of view of this commutator. In higher dimensions this commutator has to be non-zero.) Then one has to close the structure and compute  $[P_i, P_j]$  by imposing some suitable requirement. Usually one imposes the Jacobi identity (see the review in [39]) associated with the Poisson structure, but given Stueckelberg's reasoning one should

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<sup>6</sup>In three dimensions he proposes using the minimum uncertainty among all possible directions.

impose the Poisson-Malcev structure and thus relate the magnetic charge to the minimal length. This intuition makes sense in the string case because the  $H$  flux which is responsible for nonassociativity is related to the curvature of the metric by the string equations of motion [25]. The connection of non commutativity and nonassociativity and the string uncertainty principle was also investigated in [25].

### 3 Nonassociative star products and D-branes in curved backgrounds

In order to introduce the concept of nonassociative star products in string theory we follow the discussion originally presented in [23]. Note that the very important question of the consistent string background was not addressed in [23]. This crucial question was cleared up in the more recent work [25]. The basic set-up is that of open strings propagating in a curved background, following the well-known construction of noncommutative field theories in open string theory [19, 20, 21, 22, 23, 24]. The sigma model action is ( $2\pi\alpha' = 1$ )

$$S = \frac{1}{2} \int g_{ab}(X) dX^a \wedge *dX^b + \frac{i}{2} \int B_{ab}(X) dX^a \wedge dX^b \quad (24)$$

where the integration runs over the string worldsheet. By turning on the  $U(1)$  field at the boundary of the worldsheet one also has to include

$$S_b = \int F_{ab} dX^a \wedge dX^b. \quad (25)$$

By going to the Riemann normal coordinates and expanding the metric and the  $B$  field around constant backgrounds one gets, to leading order

$$S = \frac{1}{2} g_{ab} \int dX^a \wedge *dX^b + i \int \omega + \frac{i}{6} H_{abc} \int X^a dX^b \wedge dX^c \quad (26)$$

where the effective symplectic structure [23]

$$\omega_{ab}(x) = B_{ab} + F_{ab}(x) \quad (27)$$

and where the 3-form field strength is given by

$$H = dB. \quad (28)$$

To leading order, i.e. in the weakly curved backgrounds,  $H$  is constant.

The correlation function of operators inserted at the boundary in the case when  $H = 0$  are given by the well known result

$$\langle f_1 f_2 \dots f_n \rangle = \int V(\omega) dx (f_1 * f_2 * \dots * f_n), \quad (29)$$

where  $V(\omega)$  is the appropriate volume form [23] and  $*$  is the associative Weyl-Moyal-Kontsevitch star product with respect to the symplectic structure  $\omega^{-1}$

$$f * g = fg + \frac{i}{2} \omega^{ab} \partial_a f \partial_b g + \dots \quad (30)$$

The associativity is equivalent to the fact that  $\omega$  is closed,  $d\omega = 0$  [23].

The original claim found in [23] is that if one considers the formal  $g_{ab} \rightarrow 0$  limit above star product becomes nonassociative (note that the metric part  $g_{ab}$  is actually crucial for providing a consistent string background as shown in [25]). Thus formally one gets

$$f \bullet g = fg + \frac{i}{2} \tilde{\omega}^{ab} \partial_a f \partial_b g + \dots \quad (31)$$

where, for constant  $H$

$$\tilde{\omega}_{ab}(x) = B_{ab} + \frac{1}{3} H_{abc} x^c + F_{ab}. \quad (32)$$

Thus  $\tilde{\omega}$  is not closed,  $d\tilde{\omega} = H$ , and the  $\bullet$  product is not associative

$$(f \bullet g) \bullet h - f \bullet (g \bullet h) = \frac{1}{6} \tilde{\omega}^{ia} \tilde{\omega}^{jb} \tilde{\omega}^{kc} H_{abc} \partial_i f \partial_j g \partial_k h + \dots \quad (33)$$

From the world-volume point of view, this leads to non-commutativity and nonassociativity of coordinates. The commutator

$$[x^i, x^j]_{\bullet} \equiv x^i \bullet x^j - x^j \bullet x^i \quad (34)$$

is given by the following expression

$$[x^i, x^j]_{\bullet} = i \tilde{\omega}^{ij}. \quad (35)$$

Furthermore, the Jacobi identity is violated

$$[x^i, [x^j, x^k]_{\bullet}]_{\bullet} + [x^j, [x^k, x^i]_{\bullet}]_{\bullet} + [x^k, [x^i, x^j]_{\bullet}]_{\bullet} = -\tilde{\omega}^{ia} \tilde{\omega}^{jb} \tilde{\omega}^{kc} H_{abc}. \quad (36)$$

This violation of the Jacobi identity is the starting point for our discussion of the relevant nonassociative structures, Malcev algebras and their generalizations. In the context of closed string theory in the constant  $H$  background  $\tilde{\omega}_{ij} \sim H_{ijk}p^k$ ,  $p$  being the momentum [25], the resulting non-associative algebra is isomorphic to the Malcev algebra describing the motion of an electron moving in the field of a constant magnetic charge distribution [28]. As pointed out in [25] the consistency of the conformal field theory involving the constant  $H$  field crucially relies on the non-trivial gravitational background. Hence we expect the more general "magnetic" charge distributions to be described by other Malcev algebras and their generalizations. Thus the non-associative structure associated with the above violation of the Jacobi identity should be understood as a first step in the construction of a more general nonassociative structure which is in some sense "generally covariantized" in the presence of a non-trivial gravitational background [40].

## 4 Three-cocycles, nonassociativity and Malcev algebras

In this section we point out that the nonassociative structure encountered in the above violation of the Jacobi identity is well known in the quantum mechanics of three-cocycles and is associated with the mathematical concept of Malcev algebras.

The canonical example is the quantum mechanics of a non-relativistic electron moving in the background of a magnetic charge. As pointed out long time ago by Lipkin, Weisberger and Peshkin [27], the commutators of the velocities of an electron in the field of point-like magnetic monopole do not satisfy the Jacobi identity at the position of the location of the magnetic monopole.

The basic commutation relations read

$$[q_a, q_b] = 0, \quad [q_a, v_b] = i\delta_{ab} \quad (37)$$

and in particular

$$[v_a, v_b] = i\epsilon_{abc}B_c \quad (38)$$

where  $a, b, c = 1, 2, 3$ , and  $\vec{B}$  denotes the magnetic field. The commutators of the velocities yield the following Jacobian-like quantity

$$[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = -\vec{\nabla} \cdot \vec{B} \quad (39)$$

and provide a classic example of a non-vanishing three-cocycle as discussed in [26].

Following Jackiw's presentation in [26], consider how the wave-function of the charge particle transform under translations, a gauge-invariant action of which is represented by the operator ( $\hbar = 1$ )

$$U(\vec{a}) = \exp(i\vec{a} \cdot \vec{v}). \quad (40)$$

The action of the translation group on the wave-function  $\Psi(\vec{q})$  of the electron is then given by

$$U(\vec{a})\Psi(\vec{q}) = \exp(i\vec{a} \cdot \vec{v}) \exp(-i\vec{a} \cdot \vec{p})\Psi(\vec{q} + \vec{a}). \quad (41)$$

Here the momentum  $\vec{p}$  is  $\vec{p} = \vec{v} + \vec{A}$  and  $\nabla \times \vec{A} = \vec{B}$ . The three cocycle appears if one considers

$$[U(\vec{a}_1)U(\vec{a}_2)]U(\vec{a}_3)\Psi(\vec{q}) = \exp(i\alpha_3)U(\vec{a}_1)[U(\vec{a}_2)U(\vec{a}_3)]\Psi(\vec{q}). \quad (42)$$

In the case of a magnetic monopole,  $\nabla \cdot \vec{B} = 4\pi\delta(\vec{q})$ , the three-cocycle is proportional to the flux out of the tetrahedron formed from the three vectors  $\vec{a}_1, \vec{a}_2, \vec{a}_3$ , with one vertex at  $\vec{q}$ .

Thus the three-cocycle arising in this situation is a signal that the operators representing translations become nonassociative, so that the Jacobi identity fails. (Thus, operators can no longer be linear operators acting in a Hilbert space.)

It was pointed out in [41], in any formulation of the quantum mechanical magnetic monopole problem in which the coordinates and the velocities of the electron are represented by operators acting on a Hilbert space, the Jacobi identity can not be violated, since such operators are always associative. The Jacobi identity can only be violated if the "representatives" of coordinates and velocities are not operators in a Hilbert space, but belong to an intrinsically nonassociative algebra of observables. The quantum mechanical description of a non-relativistic electron in the field of a magnetic charge distribution in terms of a nonassociative algebra of observables was studied by Günaydin and Zumino [28]. They showed, in particular, that the nonassociative algebraic structure defined by velocities and coordinates of an electron in a *constant* magnetic charge distribution is that of a Malcev algebra.

We now turn our attention to a brief review of the fundamental properties of nonassociative algebras following [42, 43, 28]. The basic concept

introduced in the study of nonassociative algebraic structures is that of the associator. Given three elements  $A, B, C$  of an algebra  $\mathcal{A}$  their associator  $[A, B, C]$  is defined as

$$[A, B, C] \equiv (AB)C - A(BC). \quad (43)$$

Obviously, the associator vanishes for associative algebras. An alternative algebra is defined as an algebra in which the following identities hold

$$[A, A, B] = 0, \quad [B, A, A] = 0. \quad (44)$$

By replacing  $A$  by  $A + C$  in these formulae one also gets

$$[A, C, B] + [C, A, B] = 0 \quad [B, A, C] + [B, C, A] = 0. \quad (45)$$

Therefore, in an alternative algebra the associator  $[A, B, C]$  is an alternating function of its arguments  $A, B, C$ . The following, so-called Moufang identities, can be derived using the basic properties of the associator in an alternative algebra [42, 43, 28]

$$(ABA)C = A(B(AC)), \quad C(ABA) = ((CA)B)A, \quad (AB)(CA) = A(BC)A. \quad (46)$$

Note that

$$(ABA) = (AB)A = A(BA) \quad (47)$$

in alternative algebra.

Now, given an alternative algebra  $\mathcal{A}$  one can define an algebra  $\mathcal{A}^-$  with an anti-symmetric product (i.e. commutator). The algebra  $\mathcal{A}^-$  is not in general a Lie algebra, because the Jacobi identity is not satisfied in  $\mathcal{A}^-$ . The Jacobian is defined by

$$J(A, B, C) = [A, [B, C]] + [B, [C, A]] + [C, [A, B]] \quad (48)$$

and is proportional to the associator

$$J(A, B, C) = 6[A, B, C] \quad (49)$$

which generally does not vanish. The algebra  $\mathcal{A}^-$  is a Malcev algebra [29]. It is defined as an algebra with an anti-symmetric product  $\star$

$$A \star B = -B \star A \quad (50)$$

and a fourth order identity, called Malcev identity [29]

$$(A \star B) \star (A \star C) = ((A \star B) \star C) \star A + ((B \star C) \star A) \star A + ((C \star A) \star A) \star B. \quad (51)$$

It can be shown that the Malcev identity is equivalent to

$$J(A, B, A \star C) = J(A, B, C) \star A. \quad (52)$$

The Malcev identity is trivially satisfied in the case of Lie algebras since the Jacobian vanishes identically. Thus one can view Malcev algebras as a generalization of Lie algebras. Malcev algebras arise naturally from alternative algebras under the commutator product. One of the best known examples of an alternative algebra which is not associative is the composition algebra of octonions. The seven imaginary units of the octonions close under commutation and form a Malcev algebra which is the unique finite dimensional simple Malcev algebra, up to isomorphisms, and which is not a Lie algebra.

Returning to the example of a non-relativistic electron moving in the magnetic field it can be now shown that the Malcev identity

$$J(v_1, v_2, [v_1, v_3]) = [J(v_1, v_2, v_3), v_1] \quad (53)$$

implies that the divergence of the magnetic field,  $\nabla \cdot \vec{B}$ , must be  $q$  independent. Thus the algebra of velocities of a non-relativistic electron in the constant magnetic field is a non-commutative and nonassociative Malcev algebra. Since we have the usual canonical commutation relations between coordinates  $q_i$  and  $v_j$  the resulting Malcev algebra is infinite dimensional.

It is easy to see that the non-commutative and nonassociative algebra of coordinates found in closed string theory in the presence of a constant  $H$  background [25], as discussed at the end of section 3, is isomorphic to the algebra of velocities of a non-relativistic electron in the field of a constant magnetic charge, and therefore, it is a Malcev algebra<sup>7</sup>. However, as pointed out at the end of section 3, in the presence of a non-trivial gravitational background that is needed for the consistent conformal field theory, one needs to generalize the Malcev algebra by an appropriate “general covariantization”.

One natural generalization of the above Malcev algebra can be obtained by putting velocities and coordinates on an equal footing by introducing

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<sup>7</sup>Note that in the context of string theory  $v$ ’s (velocities) play the role of coordinates and  $q$ ’s (coordinates) of the corresponding velocities (or momenta)!

”electric” charges that are dual to the already existing “magnetic” charges. The the full ”phase space covariant” algebra with both “electric” and “magnetic charges” take the form:

$$[q_a, q_b] = -i\epsilon_{abc}E_c, \quad [q_a, v_b] = i\delta_{ab} \quad [v_a, v_b] = i\epsilon_{abc}B_c \quad (54)$$

$$[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = -\vec{\nabla} \cdot \vec{B} \quad (55)$$

$$[q_1, [q_2, q_3]] + [q_2, [q_3, q_1]] + [q_3, [q_1, q_2]] = \vec{\nabla} \cdot \vec{E}. \quad (56)$$

The above electric magnetic duality-covariant generalization of the algebra given in [28] *does not* satisfy the Malcev identity and it represents a natural generalization of the Malcev algebra discussed in [28]. Note that since this generalization requires both coordinates and velocities to be noncommuting we have to assume that both  $E$  and  $B$  fields must depend on coordinates as well as velocities and that is why the Malcev identity is not satisfied in general.

Finally, we stress that the Malcev algebra structure discussed in this section (following [28]) without the ”electric” sources is precisely the quantum version of the classical Poisson-Malcev structure discussed in section 2 in connection with Stueckelberg’s classic paper [8]. The above generalization of the algebra of [28] which includes both electric and magnetic charges corresponds to a covariant generalization of the Poisson-Malcev structure given in [33]. However our generalization given above requires that the symplectic metric in phase space given in equation (10) must not be covariantly constant. This in turn implies that the classical phase space formulation of Stueckelberg must be further generalized to phase spaces having non-vanishing ”electric” as well as ”magnetic” sources. In this more general case one has to extend the implications of a minimal length to having both a minimal momentum as well as a minimal length.

We should point out that the authors of [25] argued that the relevant structure for the central example discussed in this section of an electron moving in the field of a magnetic charge is the so-called twisted Poisson structure studied in [44]. It would be interesting to understand the connection between twisted Poisson structures and their quantization on one side and the Poisson-Malcev algebras and Malcev algebras on the other side.

In particular, given the relevance of Malcev algebras (and their classical counterparts) in the context of more general string backgrounds, it is natural to expect that Malcev algebras (and their generalizations) find their natural realization in string field theory. We will comment on this issue in the

concluding section of this paper.

## 5 Malcev Algebras, 3-forms and exceptional Lie Algebras

Since the operators acting on the Hilbert space of a quantum mechanical system are associative, any intrinsically nonassociative algebra can not be realized by such operators. This would of course not be true for any quantum mechanical system that has no Hilbert space formulation. As pointed out in the introduction there is a unique quantum mechanical system that has no formulation over an Hilbert space. This is the octonionic quantum mechanics which was shown to satisfy all the axioms of quantum mechanics within the Jordan density matrix formalism of Jordan and which has no Hilbert space realization [2]. Later the Jordan formulation of quantum mechanics was generalized to the *quadratic* Jordan formulation which extends to the octonionic quantum mechanics as well as allowing one to define quantum mechanics over finite fields [5].

The algebraic structure underlying the octonionic quantum mechanics is the exceptional Jordan algebra  $J_3^{\mathbf{O}}$  defined by  $3 \times 3$  hermitian octonionic matrices. The Jordan algebras are defined by a symmetric product

$$A \cdot B = B \cdot A \quad (57)$$

and the Jordan identity

$$A \cdot (B \cdot A^2) = (A \cdot B) \cdot A^2. \quad (58)$$

The algebra  $J_3^{\mathbf{O}}$  is the unique Jordan algebra that has no realization in terms of associative matrices with the Jordan product being one half the anti-commutator. Even though the algebra  $J_3^{\mathbf{O}}$  has no realization in terms of associative matrices it still admits an embedding into a Lie algebra that has realization in terms of associative matrices. In fact all Jordan algebras admit such an embedding into a Lie algebra. Conversely one can construct Lie algebras from Jordan algebras through what is known as Tits-Koecher-Kantor (TKK) construction [45].

Consider a 3-graded Lie algebra  $g$ :

$$g = g^{-1} \oplus g^0 \oplus g^{+1} \quad (59)$$

where  $\oplus$  denotes vector space direct sum and  $g^0$  is a subalgebra of maximal rank. We have the formal commutation relations of the elements of various grade subspaces

$$[g^m, g^n] \subseteq g^{m+n}; m, n = -1, 0, 1 \quad (60)$$

where  $g^{m+n} = 0$  if  $|m+n| > 1$ . Every simple Lie algebra with such a 3-graded structure can be constructed in terms of an underlying Jordan triple system (JTS)  $V$  via the TKK construction [45]. This construction establishes a one-to-one mapping between the grade +1 subspace of  $g$  and the underlying JTS  $V$ :

$$U_a \in g \iff a \in V. \quad (61)$$

Every such Lie algebra  $g$  admits a conjugation (involutive automorphism)  $\dagger$  under which the elements of the grade +1 subspace get mapped into the elements of the grade -1 subspace.

$$U^a = U_a^\dagger \in g^{-1}. \quad (62)$$

One then defines

$$\begin{aligned} [U_a, U^b] &= S_a^b \\ [S_a^b, U_c] &= U_{(abc)} \end{aligned} \quad (63)$$

where  $S_a^b \in g^0$  and  $(abc)$  is a triple product under which the elements of  $V$  close. Under conjugation  $\dagger$  one finds

$$\begin{aligned} (S_a^b)^\dagger &= S_b^a \\ [S_a^b, U^c] &= -U^{(bac)}. \end{aligned} \quad (64)$$

The Jacobi identities in  $g$  are satisfied if and only if the ternary product  $(abc)$  satisfies the defining identities of a JTS:

$$\begin{aligned} (abc) &= (cba) \\ (ab(cdx)) - (cd(abx)) - (a(dcb)x) + ((cda)bx) &= 0. \end{aligned} \quad (65)$$

The generators  $S_a^b$  belonging to the grade zero subspace form a subalgebra which is called the structure algebra of  $V$ :

$$[S_a^b, S_c^d] = S_{(abc)}^d - S_c^{(bad)} = S_a^{(dcb)} - S_{(cda)}^b. \quad (66)$$

The exceptional Lie algebras  $G_2, F_4$  and  $E_8$  do not admit a TKK type construction. A generalization of the TKK construction to more general triple systems was given by Kantor [46]. All finite dimensional simple Lie

algebras admit a construction over these generalized triple systems which we call Kantor triple systems (KTS). Kantor's construction of Lie algebras was generalized to a unified construction of Lie and Lie superalgebras in [47].

Kantor construction starts from the fact that every simple Lie algebra  $g$  admits a 5-grading (Kantor structure) with respect to some subalgebra  $g^0$  of maximal rank [46, 47]:

$$g = g^{-2} \oplus g^{-1} \oplus g^0 \oplus g^{+1} \oplus g^{+2}. \quad (67)$$

One associates with the grade +1 subspace of  $g$  a triple system  $V$  and labels the elements of  $g^{+1}$  subspace with the elements of  $V$  [46, 47]:

$$U_a \in g^{+1} \iff a \in V. \quad (68)$$

Every simple Lie algebra  $g$  admits a conjugation which maps the grade  $+m$  subspace into the grade  $-m$  subspace. Therefore one can also label the elements of the grade  $-1$  subspace by the elements of  $V$  :

$$U^a \equiv U_a^\dagger \in g^{-1} \iff U_a \in g^{+1}. \quad (69)$$

One defines the commutators of  $U_a$  and  $U^b$  as

$$\begin{aligned} [U_a, U^b] &= S_a^b \in g^0 \\ [U_a, U_b] &= K_{ab} \in g^{+2} \\ [U^a, U^b] &= K^{ab} \in g^{-2} \\ [S_a^b, U_c] &= U_{(abc)} \in g^{+1} \end{aligned} \quad (70)$$

where  $(abc)$  is the triple product under which the elements of  $V$  close. The remaining non-vanishing commutators of  $g$  can all be expressed in terms of the triple product  $(abc)$ :

$$\begin{aligned} [S_a^b, U^c] &= -U^{(bac)} \\ [K_{ab}, U^c] &= U_{(acb)} - U_{(bca)} \\ [K^{ab}, U_c] &= -U^{(bca)} + U^{(acb)} \\ [S_a^b, S_c^d] &= S_{(abc)}^d - S_c^{(bad)} \\ [S_a^b, K_{cd}] &= K_{(abc)d} + K_{c(abd)} \\ [S_a^b, K^{cd}] &= -K^{(bac)d} - K^{c(bad)} \\ [K_{ab}, K^{cd}] &= S_{(acb)}^d - S_{(bca)}^d - S_{(adb)}^c + S_{(bda)}^c. \end{aligned} \quad (71)$$

The Jacobi identities of  $g$  follow from the following identities [46, 47]

$$\begin{aligned} (ab(cdx)) - (cd(abx)) - (a(dcb)x) + ((cda)bx) &= 0 \\ \{(ax(cbd)) - ((cbd)xa) + (ab(cxd)) + (c(bax)d)\} - \{c \leftrightarrow d\} &= 0 \end{aligned} \quad (72)$$

which we take as the defining identities of a KTS. In general a given simple Lie algebra can be constructed in several different ways by the above method corresponding to different choices of the subalgebra  $g^0$  and different ternary algebras.

In many instances ternary algebras can be defined in terms of an underlying "binary" algebras. For example one can define a triple product over a Jordan algebra that satisfies the two conditions defining a Jordan triple system as follows:

$$(abc) \equiv a \circ (b \circ c) + c \circ (b \circ a) - (a \circ c) \circ b = (cba). \quad (73)$$

In those instances when the ternary algebras are defined by ordinary algebras, one can reverse the process of TKK or Kantor construction to define algebras from Lie algebras.<sup>8</sup> In fact this is how Jordan superalgebras were defined and classified by Kac [48]. Below we give more examples of such embeddings and discuss the case of a simple Malcev algebra of dimension seven defined by imaginary octonions.

The division algebras  $\mathbf{A} = \mathbf{R}, \mathbf{C}, \mathbf{H}$  and  $\mathbf{O}$  and their tensor products with each other define KTS's under the ternary product

$$(abc) = a \cdot (\bar{b} \cdot c) + c \cdot (\bar{b} \cdot a) - b \cdot (\bar{a} \cdot c) \quad (74)$$

where  $a, b, c$  are elements of  $\mathbf{A} \times \mathbf{A}'$  and the bar denotes conjugation in the underlying division algebras. Using the Kantor construction above one obtains the Lie algebras of the famous Magic Square. In Table 1. we list these algebras and the corresponding Lie algebras.

Consider the construction of the exceptional Lie algebra  $F_4$  over the division algebra  $\mathbb{O}$  of octonions. The triple product in that case is  $(abc) = a \cdot (\bar{b} \cdot c) + c \cdot (\bar{b} \cdot a) - b \cdot (\bar{a} \cdot c)$ . Let  $e_0$  be the identity element of  $\mathbb{O}$  and  $e_A$  be the seven imaginary units. Consider the elements  $U_{e_A}$  of the Lie algebra  $F_4$  belonging to grade +1 space labelled by  $e_A$ . Then define the product  $\star$  between two elements of grade +1 space via the double commutator

$$U_{e_A} \star U_{e_B} \equiv [[U_{e_A}, U_{e_B}], U^{e_0}] = U_{[e_A, e_B]}. \quad (75)$$

Then under this product the grade +1 elements  $U_{e_A}$  of  $F_4$  form a Malcev algebra of dimension seven.

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<sup>8</sup> Similarly, one can give a vertex operator construction of underlying algebras and triple systems and their affine extensions starting from affine Lie algebras and super-algebras[49].

Table 1:

$\mathbf{A} \times \mathbf{A}'$	$\mathbf{R}$	$\mathbf{C}$	$\mathbf{H}$	$\mathbf{O}$
$\mathbf{R}$	$SO(3)$	$SU(3)$	$USp(6)$	$F_4$
$\mathbf{C}$	$SU(3)$	$SU(3)^2$	$SU(6)$	$E_6$
$\mathbf{H}$	$USp(6)$	$SU(6)$	$SO(12)$	$E_7$
$\mathbf{O}$	$F_4$	$E_6$	$E_7$	$E_8$

### 5.1 Exceptional Lie algebras and anti-symmetric tensors of rank three

Since the three form field strengths play a fundamental role in the results presented in this paper as well as in [25] let us investigate their algebraic properties. Clearly one can not define a binary product over the space of tensors of rank three. However one can define a triple product among them such that they close under it. In fact, exceptional Lie algebras of the  $E$ -series can be constructed in a unified manner over triple systems defined by antisymmetric tensors of rank three in various dimensions [46]. Let  $x_{ijk}, y_{ijk}, z_{ijk}, (i, j, \dots = 1, 2, \dots, n)$  be totally antisymmetric tensors of rank three in  $n$  dimensions. Define a ternary product among such tensors as

$$(xyz)_{ijk} = y_{imn}x_{pmn}z_{pjk} + y_{pjn}x_{pmn}z_{imk} + y_{pmk}x_{pmn}z_{ijn} - \frac{1}{3}y_{mnp}x_{mnp}z_{ijk}. \quad (76)$$

For  $n = 6, 7$  this triple product satisfies the defining conditions of a KTS and lead to the construction of  $E_6$  and  $E_7$ , respectively.

$$\begin{aligned} E_6 &= T^- \oplus T_{ijk} \oplus T_j^i \oplus T^{ijk} \oplus T^+ \\ 78 &= \bar{1} \quad 20 \quad U(6) \quad 20 \quad 1 \end{aligned} \quad (77)$$

$$\begin{aligned} E_7 &= T_i \oplus T_{ijk} \oplus T_j^i \oplus T^{ijk} \oplus T^i \\ 133 &= \bar{7} \quad \bar{35} \quad U(7) \quad 35 \quad 7 \end{aligned} \quad (78)$$

For  $n = 5$  and  $n = 4$  the above triple product satisfies the defining identities of a JTS and lead to the construction of  $E_5 = SO(10)$  and  $E_4 = SU(5)$ , respectively.

$$\begin{aligned} E_5 &= T_{ijk} \oplus T_j^i \oplus T^{ijk} \\ 45 &= \bar{10} \quad U(5) \quad 10 \end{aligned} \quad (79)$$

$$\begin{array}{rcl} E_4 = & T_{ijk} \oplus & T_j^i \oplus T^{ijk} \\ 24 = & \bar{10} & U(5) \quad 10 \end{array} \quad (80)$$

For  $n = 8$  the generalized Kantor construction leads to the construction of the largest exceptional Lie algebra  $E_8$  with a seven graded structure.

$$\begin{array}{rcl} E_8 = & T_i \oplus & T^{ij} \quad T_{ijk} \oplus \quad T_j^i \oplus \quad T^{ijk} \oplus \quad T_{ij} \quad T^i \\ 248 = & \bar{8} & 28 \quad \bar{56} \quad U(8) \quad 56 \quad \bar{28} \quad 8 \end{array} \quad (81)$$

In nine dimension the generalized Kantor construction leads to the construction of the affine Lie algebra  $E_9$ . It is tantalizing to speculate that a further generalization of the Kantor construction leads to the construction of the hyperbolic Lie algebras  $E_{10}$  and  $E_{11}$  in terms of antisymmetric tensors in ten and eleven dimensions, respectively.

One can reverse the above construction of exceptional Lie algebras and define the triple systems of antisymmetric tensors of rank three in dimensions  $d < 9$  corresponding to a subspace of  $E_d$  involving double commutators as was done for the simple Malcev algebra of dimension seven.

## 6 Simple Malcev algebra $\mathbb{O}^-$ and related geometric structures

In this section we point out that the nonassociative algebraic structures discussed above appear naturally in the context of certain special geometries. The relation between the 6-manifolds with the  $G_2$  invariant almost complex structures and the simple Malcev algebra  $\mathbb{O}^-$  was investigated in the mathematics literature long time ago [50, 51, 52, 53, 54]. These geometries can be naturally associated with new backgrounds of string theory in which a non-constant  $H$  field is turned on.

In particular, as discussed in detail in [55], the almost complex structure and the associated torsion tensor on  $S^6$  are intimately related to octonions. Here we review the basic facts about it closely following [55]. An almost complex structure of an even dimensional manifold  $M$  is a mixed tensor  $F_k^i$  which satisfies

$$F_k^i F_j^k = -\delta_j^i \quad (82)$$

The torsion of an almost complex structure is defined as [51]

$$\tau_{jk}^i \equiv \frac{1}{2}(A_{mj}^i F_k^m - A_{mk}^i F_j^m) \quad (83)$$

where  $A_{jk}^i$  is given by the following expression

$$A_{jk}^i \equiv \frac{1}{2}(\nabla_j F_k^i - \nabla_k F_j^i) \quad (84)$$

where  $\nabla_i$  is a covariant derivative on  $M$  with respect to some symmetric affine connection. The symmetry of the affine connection implies that

$$A_{jk}^i \equiv \frac{1}{2}(\partial_j F_k^i - \partial_k F_j^i) \quad (85)$$

Note that the almost complex structure  $F_i^j$  is integrable to a complex structure provided the torsion tensor  $\tau_{ij}^k$  vanishes [51, 50].

As emphasized in [55] an almost complex structure of  $S^6$  can be defined using the imaginary Cayley numbers [51, 50, 52], by thinking of  $S^6$  as a hypersurface in the space of imaginary octonions, also known as Cayley space  $I^7$  [53]. A vector  $X$  in  $I^7$  can be represented as ( $A = 1, \dots, 7$ )

$$X = X^A e_A \quad (86)$$

where  $e_A$  are the imaginary units of octonions [11]. The natural scalar product of two vectors  $X$  and  $Y$  in  $I^7$  reads as follows

$$(X, Y) \equiv \frac{1}{2}(\bar{X}Y + X\bar{Y}) = X^A Y_A \quad (87)$$

where bar denotes octonion conjugation  $e_A \rightarrow -e_A$ . The natural cross product of two vectors is given by the commutator

$$X \wedge Y \equiv \frac{1}{2}[XY - YX] = -Y \wedge X. \quad (88)$$

Under this product the space  $I^7$  becomes the unique simple finite dimensional Malcev algebra defined by the imaginary octonions.

Let us, following [55], denote the unit normal to  $S^6$  as  $\hat{n}$  and choose a set of basis vectors  $\hat{e}_i$  in the tangent space so that the metric  $g_{ij}$  on  $S^6$  is

$$g_{ij} = (\hat{e}_i, \hat{e}_j). \quad (89)$$

Then the almost complex structure on  $S^6$  can be defined via

$$\hat{e}_i \wedge \hat{n} = F_i^j \hat{e}_j. \quad (90)$$

From the properties of the cross product one can verify that indeed  $F_k^i F_j^k = -\delta_j^i$  [55]. Using the cross product one can also define a mixed tensor  $T_{ij}^k$

$$\hat{e}_i \wedge \hat{e}_j = -F_{ij} \hat{n} + T_{ij}^k \hat{e}_k. \quad (91)$$

Note that  $T_{ijk} = T_{ij}^m g_{mk}$  is completely antisymmetric in its indices

$$T_{ijk} = (\hat{e}_i \wedge \hat{e}_j, \hat{e}_k) = -T_{jik} = T_{kij} \quad (92)$$

and that  $T_{ijk} T^{ljk} = 4\delta_i^l$ . By utilizing the equations of Gauss and Weingarten for the covariant derivatives  $\nabla_i$

$$\nabla_i \hat{e}_j = H_{ij} \hat{n}, \quad \nabla_i \hat{n} = -H_i^j \hat{e}_j, \quad (93)$$

where  $H_{ij}$  is the second fundamental tensor, and the Gauss-Codazzi equations for the curvature  $R_{ijkl} = H_{il} H_{jk} - H_{jl} H_{ik}$  and  $\nabla_k H_{ji} = \nabla_j H_{ki}$ , one can show that for  $S^6$  [55]

$$H_{ij} = \lambda g_{ij}. \quad (94)$$

By covariant differentiation of  $\hat{e}_i \wedge \hat{n} = F_i^j \hat{e}_j$  one finds

$$\nabla_j F_i^k = H_j^m T_{mi}^k. \quad (95)$$

Furthermore by covariantly differentiating  $\hat{e}_i \wedge \hat{e}_j = -F_{ij} \hat{n} + T_{ij}^k \hat{e}_k$  one obtains on  $S^6$

$$\nabla_k T_{ij}^k = -4\lambda F_{ij} \quad (96)$$

as well as

$$\nabla_k F_{ij} = T_{ijk}. \quad (97)$$

The almost complex structure  $F_{ij}$  is covariant under the automorphism group  $G_2$  of octonions and the relevant nonassociative algebraic structure is the Malcev algebra  $\mathbb{O}^-$  defined by the seven imaginary units of octonions.

In the context of our note, the almost complex structure  $F_{ij}$  can obviously play the role of the  $B$  field and the torsion  $T_{ijk}$  of the associated non-vanishing  $H$  field. As such they correspond to a background with non-constant  $H$  and  $B$  fields defined by an underlying Malcev algebra <sup>9</sup>.

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<sup>9</sup> The coset space  $G_2/SU(3)$  is one of the spaces considered in [56] on compactifications of superstrings on six dimensional Ricci flat coset spaces and, more recently, in the work of [57] on backgrounds that lead to four dimensional anti-de Sitter space times.

## 7 Discussion: A generalized AdS/CFT-like duality

In this paper we first reviewed the work of Stueckelberg on the generalization of Poisson brackets in classical mechanics while preserving Boltzmann's H-theorem that, in general, violate the Jacobi identity. We proposed that the nonlinear extension of the corresponding quantum mechanics envisaged by Stueckelberg, which involves the fundamental length, should be formulated as a nonassociative extension of quantum mechanics. We found that the relevant classical structure is the Poisson-Malcev algebra and its generalizations.

In the context of string theory the nonassociative algebraic structures appear in the description of closed strings in curved backgrounds [25]. We pointed out that under certain assumptions these nonassociative structures coincide with nonassociative Malcev algebras which naturally appear in the quantum mechanics of systems with non-vanishing three-cocycles, such as an electric point particle moving in the field of a magnetic charge. We also generalized the Malcev algebra of a point particle moving in the field of a magnetic charge to include both electric and magnetic charges.

In recent work [25] the role of nonassociativity was explored in closed string theory involving non-trivial gravitational backgrounds. One of the hallmarks of quantum gravity is holography. Given the holographic formulation of string theory in asymptotically AdS backgrounds, it is natural to ask whether a generalized AdS/CFT-like dictionary [58] can be formulated involving nonassociative structures in closed string theory, by relating them to noncommutative yet associative structures in open string theory. (As pointed out in section 4, the nonassociative structures can in general be embedded into Lie algebras. The question is whether this purely algebraic relation between nonassociative and associative algebraic structures can be extended to include dynamics.)

To address this possibility, we first note that the failure of the Jacobi identity of the form discussed in this paper is also known to occur in purely cubic open string field theory [13]. This associativity anomaly can be demonstrated by an oscillator calculation in purely cubic string theory. The technical reason for the appearance of this associativity anomaly can be understood as follows: one can think of operators appearing in open string field theory in terms of infinite dimensional matrices. For a class of infinite dimensional matrices single sums appearing in the usual product of two matrices are

absolutely convergent, yet double sums, appearing in the product of three matrices are not absolutely convergent, leading to associativity anomalies.

It is claimed in the literature that this nonassociativity in purely cubic open string field theory is responsible for the appearance of closed string states [13]. In particular, Strominger [14, 15], has proposed a gauge invariant cubic action describing bosonic closed string field theory, in which the basic dynamical objects are open strings. The action is given by the associator for the string field product, which due to the associativity anomaly is non-vanishing.

Very roughly, Strominger’s closed string field theory action looks like its open string field theory counterpart, i.e. it is cubic

$$S_c(\Psi) = \int \Psi \times (\Psi \times \Psi) \quad (98)$$

except for the nonassociative nature of the  $\times$  product. The  $\Psi$  field is also of the “open string” type. The open string field theory of Witten [35, 36] can be also written in a purely cubic form (see [34, 37])

$$S_o(\Phi) = \int \Phi * \Phi * \Phi \quad (99)$$

where  $*$  is a non-commutative but associative product and where  $\Phi$  is an open-string field. Given the fact that the Laplace transform of an exponential of a cubic (i.e. the Airy function) in the WKB limit becomes an exponent of a square root of a cubic,  $\int dx \exp(tx - x^3) \sim \exp(-t^{3/2})$ , (where  $t$  obviously scales as  $x^2$ ) one is tempted to conjecture the following AdS/CFT-like “holographic” dictionary between the generating functional of the cubic open string field theory at large coupling and its cubic closed string field theory counterpart

$$\langle \exp(\int J\Phi) \rangle_o \equiv Z[J] \rightarrow \exp[-S_c(\Psi)]. \quad (100)$$

Here  $\Psi \rightarrow J$  in the region of the closed string Hilbert space on which the 3-cocycle anomaly vanishes. (Note that  $\Psi \sim J$  scales as  $\Phi^2$ ). Thus the on-shell action for Strominger’s nonassociative cubic closed string field theory would compute the generating functional for Witten’s noncommutative cubic open string field theory at large coupling. In this sense this would be a natural non-geometric generalization of the AdS/CFT-like dictionary<sup>10</sup>.

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<sup>10</sup>This conjecture is, in some sense, made more plausible by the existence of the Legendre

Given the robust nature of Malcev algebras (and their generalizations), as discussed in sections 4 and 5 of this paper, and given the fact that associative anomalies do appear in the context of string field theory, it is natural to conjecture that the relevant algebraic structure represented by the non-associative  $\times$  product in Strominger's cubic formulation of closed string theory is of a Malcev algebra type<sup>11</sup>. Its explicit realization is an open and fascinating fundamental question. Nevertheless, we hope that the discussion presented in this article makes a compelling case for the fundamental role of nonassociativity in string theory.

**Acknowledgements:** This paper had a 12 year long gestation. We benefited from discussions with R. Gopakumar, A. Strominger and B. Pioline during the initial period. More recently we benefited from discussions with L. Freidel, R.G. Leigh, M. Kruszenski and Sung-Sik Lee and especially, D. Lüst. We also wish to acknowledge the kind hospitality of Harvard University, Caltech, USC, the Institute for Advanced Study at Princeton and the Miami Winter conference where parts of this work were initiated and developed. The work of M. G. was supported in part by the National Science Foundation under Grant Number PHY-1213183 and PHY-0855356. The work of D. M. was supported in part by the US Department of Energy under grant number DE-FG03-84ER40168.

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invariant cubic forms [6, 59, 60, 61]. Remarkably, the C-tensors of maximal supergravity as well as the exceptional supergravity define cubic forms that are Legendre invariant [60].

<sup>11</sup>The closed string fields might be realized in the context of the quadratic Jordan formulation.

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